University of Idaho
High School Mathematics Competition 2024

Division II Solutions

Division II, Problem 1

Adam and Bob have been hired to type some old handwritten manuscripts into a computer. Adam would take 16 hours to do the whole job by himself, while Bob would take 12 hours to do the whole job by himself. Suppose Adam and Bob worked together for 6 hours, and then Bob left Adam to finish by himself. How much longer did Adam work for to finish the job? Assume that they don't distract each other or duplicate work and each types at the same rate they would if they were typing by themselves.

Solution: Since Bob takes 12 hours, he finishes half the job in 6 hours. This means Adam does half the jobs, and it takes him 8 hours to do so. Hence, Adam takes 2 more hours to finish the job.

Division II, Problem 2

Find the smallest integer n with $n > 2024$ such that the equation $x^2 + x - n = 0$ has two integer solutions.

Solution: Let $k = n - 2024$. We want to find the smallest positive integer k such that

$$x^2 + x - (k + 2024) = 0$$

has two integer solutions.

By the quadratic formula, the solutions of this quadratic equation are

$$x = \frac{-1 \pm \sqrt{4k + 8097}}{2}.$$

This means that both solutions are integers if and only if $\sqrt{4k + 8097}$ is an odd integer, or equivalently if $4k + 8097$ is a square of an odd positive integer. So, we need to find the smallest odd positive integer e such that $e^2 - 8097$ is a positive integer divisible by 4.

Note that the smallest square of a positive integer, which is bigger than 8097, is $8100 = 90^2$. Thus, the integer e must be bigger than 90.

If $e = 91$, then $e^2 - 8097 = 184$, which is divisible by 4. Thus,

$$k = \frac{e^2 - 8097}{4} = \frac{184}{4} = 46,$$

and hence $n = k + 2024 = 46 + 2024 = 2070$ is the smallest positive integer satisfying the desired conditions.

Division II, Problem 3

What is the largest positive integer c such that there are NO positive integers a and b with

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}?$$

Solution: Multiply both sides of the equation by 210 to get $15a + 14b = c$. If $c = 14 \times 15$, then there are no positive integer solutions to $14 \times 15 = 15a + 14b$ as $\gcd(14, 15) = 1$, so the a must be divisible by 14 and b must be divisible by 15, which is impossible if a and b are both positive. Hence, the largest c must be at least 210.

If $c > 14 \times 15$, we can find the quotient and remainder upon dividing c by 14 and write $c = 14p + q$ for $p \geq 15$ and $0 \leq q \leq 13$.

Case I: If $q > 0$, we can express $c = 14p + q(15-14) = 15q + 14(p-q)$. So, $a = q$, $b = p - q$ solve the equation.

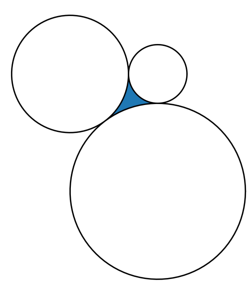
Case II. If $q = 0$, then $p > 15$. Find the quotient and remainder upon dividing p by 15 to write $p = 15k + r$ with $0 \leq r \leq 14$. If $r > 0$. We can express $c = 14p = 14 \times (15k + r) = 15(14k) + 14r$. So, $a = 14k$ and $b = r$ solve the equation.

If $r = 0$, then we must have $k > 1$. Thus, $c = 14 \times 15k = 15 \cdot 14 + 14 \cdot (15(k-1))$. So, $a = 14$, $b = 15(k-1)$ solve the equation.

Hence there is a solution whenever $c > 210$, so $c = 210$ is the largest number without a solution.

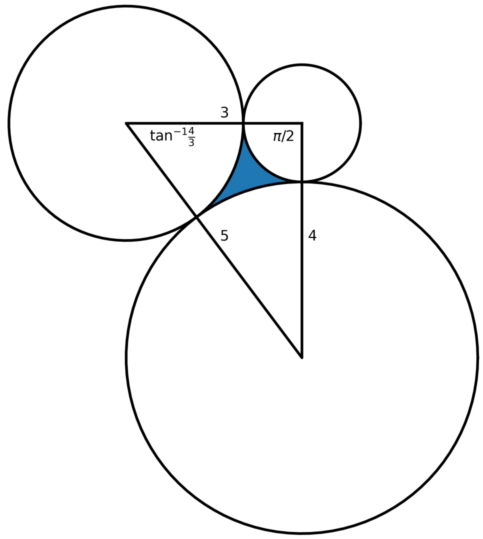
Division II, Problem 4

Three non-overlapping disks of radii 1, 2, and 3 are placed so that they are all tangent to each other. What is the area of the region between the three disks? (This is the shaded region in the diagram.)



Solution: Connecting the centers of the three disks, we obtain a triangle of side lengths $1+2$, $1+3$ and $2+3$, as shown in the figure below. The area of this right triangle is 6. The three angles of the triangle are $\tan^{-1} 3/4$, $\tan^{-1} 4/3$ and $\pi/2$ (radians). The areas of the intersection of this triangle with the three disks are respectively $\frac{3^2}{2}\tan^{-1} 3/4$, $\frac{2^2}{2}\tan^{-1} 4/3$ and $\frac{\pi}{4}$. Thus, the area of the shape surrounded by the three disks is

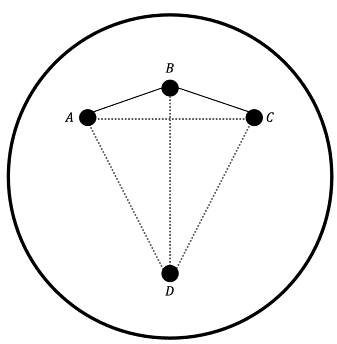
$$6 - \frac{\pi}{4} - 2\tan^{-1}(4/3) - \frac{9}{2}\tan^{-1}(3/4).$$



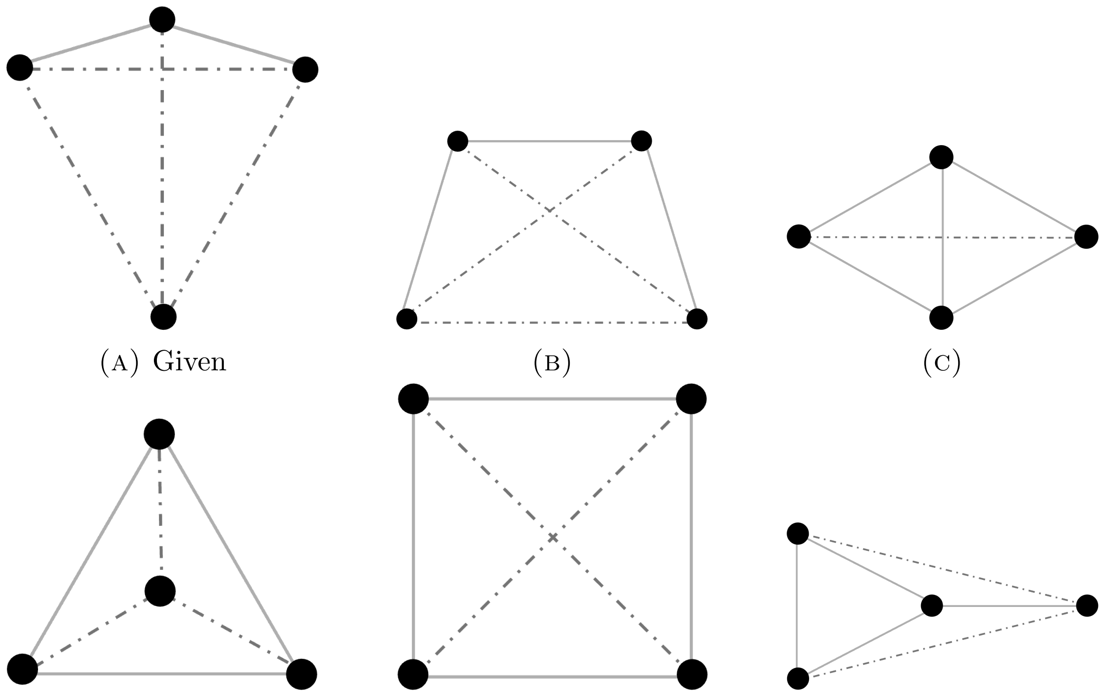
Division I, Problem 5

You wish to place four candles on a cake so that there are only 2 different distances between any 2 candles. How many different ways are there for you to do this? Make sure to justify that there are only 2 different distances for way you discover. (Two ways are considered the same if they are similar (in the geometry sense) to each other.) You do NOT need to explain why you have found all the possibilities.

One way of placing the 4 candles is shown in the picture: segments $\overline{AB}$ and $\overline{BC}$ have one length while segments $\overline{AD}$, $\overline{BD}$, $\overline{CD}$, and $\overline{AC}$ all share a second length.



Solution:



(A) Given (B) (C) (D) (E) (F)

FIGURE 1. All possible configurations for the 4 points with 2 distance problem.

There are 6 possible configurations for the four candles such that there are only 2 distances between any 2 candles. All six configurations, including the example configuration are given in Figure 1.

Justification for each configuration:

For configuration 1b, a potential justification is that this is simply 4 points of a regular pentagon. Therefore, the line segments representing the vertices will all be an equivalent length and the remaining lengths will simply be the diagonals of the regular pentagon, which will all have equivalent length.

For configuration 1c, a potential justification is that this configuration is formed by joining two equilateral triangles along a common edge. Therefore, all exterior sides of the configuration will be equivalent, along with the shared base of the triangles. Then the second length is simply the connection of the two furthest points.

For configuration 1d, a potential justification is that the configuration is formed through the creation of an equilateral triangle, which by definition must have equivalent side lengths (which will be the first length). Then the fourth point is then placed at the centroid of the equilateral triangle, where the length from the centroid to each of the vertices is the same, and this then represents the second length.

For configuration 1e, a potential justification is that the configuration is formed through the creation of square. The first length is equivalent to the square side length. Then the second length is the diagonals of the square (since the diagonals have equal length).

For configuration 1f, a potential justification is that the configuration is formed through the creation of an equilateral triangle, where the first length represents the side length of the triangle. Then the fourth point is placed along the perpendicular bisector of one of sides (which we will call the base), exactly 1 triangle side length above the vertex opposite the base. The second length then are the sides of the isosceles triangle made by the fourth point and the base.

Division II, Problem 6

Solve

$$2^x - 2^{6-x} - 12 < 0.$$

Solution: Multiplying both sides of the inequality by 2^x gives rise to

$$(2^x)^2 - 64 - 12 \cdot 2^x < 0. \tag{1}$$

Since $(2^x)^2 - 64 - 12 \cdot 2^x = (2^x + 4)(2^x - 16)$, inequality (1) holds if and only if $0 < 2^x < 16$. Thus, the desired inequality holds if and only if we get $x < 4$.

Division II, Problem 7

What is the volume of the largest hemisphere that can be hidden inside a circular cone of radius 5 and height 12? You may assume the bottom of the hemisphere is (part of) the bottom of the cone, and the hemisphere can touch but not pass through the cone.

Solution: Let A be the vertex of the cone and O be the center of the circular base of the cone. Pick a point B on the boundary of the base of the cone. Consider the right $\triangle ABO$. The side lengths of the triangle are 5, 12, 13. Thus the area is 30. Let C be the point on AB at which $AB \perp OC$. Then C is the point at which AB is tangent to the hemisphere. Thus, OC is the radius of the hemisphere. Since OC is also the height of $\triangle ABO$ that is perpendicular to AB, we have $OE = \frac{2 \times \text{area}(\triangle ABO)}{|AB|} = \frac{60}{13}$. Hence, the volume of the hemisphere is $\frac{2}{3}\pi \left(\frac{60}{13}\right)^3$.

Division I, Problem 8

Zoe plays the following one player game. On the table are 12 envelopes, numbered $1, \ldots, 12$, and the envelope numbered n has $\$n$ inside. On each turn, Zoe can take any envelope as long as that envelope has at least one other envelope labeled by one of its factors still on the table. Then all of the factors of the envelope taken are removed. Zoe then takes another turn, continuing until she can no longer take any turns (because none of the envelopes on the table have any of their factors other than themselves on the table). What is the maximum amount of money Zoe can get?

For example, Zoe can start by taking envelope 6. Once she does so, envelopes 1, 2, and 3 are also removed from the table. On the next turn, Zoe cannot take envelopes 4, 5, 7, 9, or 11 because they have no factors on the table. If Zoe takes envelope 10, then envelope 5 is also removed, and then Zoe has a choice of taking 8 or 12, either of which would end the game.

Solution: The maximum amount of money that Zoe can get is \$50. This is achieved through the selection of the following series of envelopes, as shown in Table 1.

Turn	Zoe's Chosen Envelope	Factor Envelopes Removed
1	11	1
2	10	5,2
3	9	3
4	8	4
5	12	6
Leftover	-	7

TABLE 1. Envelopes taken by Zoe to maximize the amount of money she gets

These turns can be somewhat reordered. For example, turn 2 and turn 3 can be switched since 10 and 9 have no common factors (since the \$1 envelope was already taken). Therefore, the solution is correct so long as the overall amount of money that Zoe has earned is \$50 and the choices are consistent.

The reason why this is the maximum amount of money is that Zoe may only select one prime number during this game, as the sole envelope factor that will be removed for a prime number is the \$1 envelope. Zoe should select the largest prime between 1 and 12, which is 11. This needs to be selected first otherwise any other envelope will remove the \$1 factor and she can no longer select this envelope. So to maximize her total amount of money, envelope 7 cannot be selected.

Therefore, there are actually 11 envelopes under consideration, and for each envelope that Zoe selects, one must go away, this means that the maximum number of turns she can take is 5. Therefore, select the 5 greatest envelopes (12, 11, 10, 9, 8) and this would yield the maximal amount of money, and Table 1 details how to select the 5 envelopes over the course of the game.